# ON THE STABILITY OF PERIODIC SOLUTIONS OF NON-SELF-CONTAINED QUASI-LINEAR SYSTEMS WITH TWO DEGREES OF FREEDOM

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Conditions are obtained for the asymptotic stability of the periodic solutions of non-self-contained quasilinear systems with two degrees of freedom in the case of principal resonance with one resonant frequency for simple and double roots of the equation for the fundamental amplitudes.

**1.** We consider the oscillatory system

$$x^{\cdot\cdot} + k^2 x = f^{(1)}(t) + \mu F^{(1)}(t, x, x^{\cdot}, y, y^{\cdot}, \mu)$$
$$y^{\cdot\cdot} + \omega^2 y = f^{(2)}(t) + \mu F^{(2)}(t, x, x^{\cdot}, y, y^{\cdot}, \mu) \qquad (1.1)$$

Here $f^{(1)}$ and $f^{(2)}$ are continuous functions of period $2\pi$, satisfying the conditions for the existence of periodic solutions of the generating system ($\mu = 0$) with the same period; $F^{(1)}$ and $F^{(2)}$ are analytic functions with respect to the variables $x, x^{\cdot}, y, y^{\cdot}, \mu$, and are continuous periodic functions of $t$ with period $2\pi$. The quantity $\mu$ is a small parameter, $k$ is an integer, $\omega$ is a noninteger. The generating solution of period $2\pi$ depends on two arbitrary constants $A_0$ and $B_0$

$$x_0(t) = A_0 \cos kt + B_0 k^{-1} \sin kt + f_0^{(1)}(t), \qquad y_0(t) = f_0^{(2)}(t) \qquad (1.2)$$

Here $f^{(1)}{}_0, f^{(2)}{}_0$ is a particular solution of period $2\pi$ of the system (1.1) when $\mu = 0$.

The initial conditions for system (1.1) are taken in the form [1]

$$x(0) = f_0^{(1)}(0) + A_0 + \beta_1, \qquad y(0) = f_0^{(2)}(0) + \psi_1$$
$$x^{\cdot}(0) = f_0^{(1)\cdot}(0) + B_0 + \beta_2, \qquad y^{\cdot}(0) = f_0^{(2)\cdot}(0) + \psi_2 \qquad (1.3)$$

Here $\beta_1$, $\beta_2$ are functions of $\mu$ vanishing for $\mu = 0$; $\psi_1$ and $\psi_2$ are analytic functions of $A_0 + \beta_1$, $B_0 + \beta_2$ and $\mu$ also vanishing for $\mu = 0$.

These functions may be represented by the series

$$\psi_i = \sum_{n=1}^{\infty} \left[\Psi_{in} + \frac{\partial \Psi_{in}}{\partial A_0}\beta_1 + \frac{\partial \Psi_{in}}{\partial B_0}\beta_2 + \ldots\right]\mu^n \qquad (i = 1,2) \qquad (1.4)$$

The particular solution of system (1.1) with initial conditions (1.3), turned into the generating solution (1.2) when $\mu = 0$, can be written as[1]

$$x(t) = f_0^{(1)}(t) + (A_0+\beta_1)\cos kt + \frac{B_0+\beta_2}{k}\sin kt + \tag{1.5}$$
$$+\sum_{n=1}^{\infty}\left[C_n^{(1)}(t) + \frac{\partial C_n^{(1)}(t)}{\partial A_0}\beta_1 + \frac{\partial C_n^{(1)}(t)}{\partial B_0}\beta_2 + \ldots\right]\mu^n$$
$$y(t) = f_0^{(2)}(t) + \sum_{n=1}^{\infty}\left[C_n^{(2)}(t) + \frac{\partial C_n^{(2)}(t)}{\partial A_0}\beta_1 + \frac{\partial C_n^{(2)}(t)}{\partial B_0}\beta_2 + \ldots\right]\mu^n$$

Let us note that the derivatives with respect to $A_0$ and $B_0$ are complex derivatives taken with due regard to the dependencies of $C_n^{(1)}(t)$ and $C_n^{(2)}(t)$ on $\psi_1$ and $\psi_2$ which in turn depend on $A_0$ and $B_0$ .

The functions $C_n^{(1)}(t)$ and $C_n^{(2)}(t)$ are determined by Formulas [1]

$$C_n^{(1)}(t) = \frac{1}{k}\int_0^t F_n^{(1)}(\tau)\sin k(t-\tau)\,d\tau \tag{1.6}$$

$$C_n^{(2)}(t) = \Psi_{1n}\cos\omega t + \frac{\Psi_{2n}}{\omega}\sin\omega t + \frac{1}{\omega}\int_0^t F_n^{(2)}(\tau)\sin\omega(t-\tau)\,d\tau$$

Here

$$F_n^{(i)}(\tau) = \frac{1}{(n-1)!}\left[\frac{d^{n-1}F^{(i)}(\tau)}{d\mu^{n-1}}\right]_{\beta_1=\beta_2=\psi_1=\psi_2=\mu=0}$$

are the complete partial derivatives of the functions $F^{(i)}$ with respect to $\mu$ .

For example,

$$F_1^{(i)}(t) = (F^{(i)})_0, \qquad F_2^{(i)}(t) = (F_x^{(i)})_0\, C_1^{(1)}(t) + (F_{x^\cdot}^{(i)})_0\, C_1^{(1)\cdot}(t) +$$
$$+ (F_y^{(i)})_0\, C_1^{(2)}(t) + (F_{y^\cdot}^{(i)})_0\, C_1^{(2)\cdot}(t) + (F_\mu^{(i)})_0 \qquad \text{etc.} \tag{1.7}$$

Here and what follows the subscript 0 signifies that in the functions within the parentheses the $x$, $x^\cdot$, $y$, $y^\cdot$, $\mu$ should be replaced by $x_0$, $x_0^\cdot$, $y_0$, $y_0^\cdot$, 0 from (1.2); $F_x^{(i)} = \partial F^{(i)}/\partial x$ etc. In order that the solution (1.5) be periodic with period $2\pi$ , it is necessary and sufficient that the following four Poincaré conditions for periodicity [2] be satisfied:

$$x(2\pi) - x(0) = 0, \quad y(2\pi) - y(0) = 0, \quad x^\cdot(2\pi) - x^\cdot(0) = 0, \quad y^\cdot(2\pi) - y^\cdot(0) = 0$$

From these periodicity conditions we can find:

1) the amplitudes $A_0$ and $B_0$ as solutions of the amplitude equations

$$C_1^{(1)}(2\pi) = 0, \qquad C_1^{(1)\cdot}(2\pi) = 0 \tag{1.8}$$

2) the quantities $\beta_1$ and $\beta_2$ in the form of series in $\mu$ or $\mu^{\frac{1}{2}}$ when Equations (1.8) have double roots,

$$\beta_1 = \sum_{n=1}^{\infty} A_{n/2}^{(r)}\mu^{n/2}, \qquad \beta_2 = \sum_{n=1}^{\infty} B_{n/2}^{(r)}\mu^{n/2} \qquad (r = 1, 2) \tag{1.9}$$

Here the first nonzero coefficients $A_{n/2}^{(r)}$ and $B_{n/2}^{(r)}$ are determined from quadratic equations, while the rest are determined from linear systems of equations with nonzero determinants in the unknowns [1 and 3].

3) the coefficients $\Psi_{1n}$ and $\Psi_{2n}$ from Equations [1]

$$\Psi_{1n}(\cos 2\pi\omega - 1) + \frac{\Psi_{2n}}{\omega}\sin 2\pi\omega + \frac{1}{\omega}\int_0^{2\pi} F_n^{(2)}(\tau)\sin\omega(2\pi-\tau)\,d\tau = 0$$
$$-\omega\,\Psi_{1n}\sin 2\pi\omega + \Psi_{2n}(\cos 2\pi\omega - 1) + \int_0^{2\pi} F_n^{(2)}(\tau)\cos\omega(2\pi-\tau)\,d\tau = 0 \tag{1.10}$$

Thus, to each simple root of the equations for the fundamental amplitudes there corresonds one periodic solution of (1.1) in the form of the series (1.5) in integer powers of $\mu$ (all the coefficients with fractional indices in (1.9) equal to zero); to each double root there correspond two periodic solutions of (1.1) in the form of series in $\mu$ or $\mu^{\frac{1}{2}}$

$$x^{(r)}(t) = \sum_{n=0}^{\infty} x_{n/2}^{(r)}(t)\,\mu^{n/2}, \qquad y^{(r)}(t) = \sum_{n=0}^{\infty} y_{n/2}^{(r)}(t)\,\mu^{n/2} \tag{1.11}$$

where

$$x_{1/2}^{(r)}(t) = A_{1/2}\cos kt + \frac{B_{1/2}}{k}\sin kt, \quad y_{1/2}^{(r)}(t) = 0$$

$$x_1^{(r)}(t) = A_1\cos kt + \frac{B_1}{k}\sin kt + C_1^{(1)}(t), \quad y_1^{(r)}(t) = C_1^{(2)}(t)$$

$$x_{3/2}^{(r)}(t) = A_{3/2}\cos kt + \frac{B_1}{k}\sin kt + \frac{\partial C_1^{(2)}(t)}{\partial B_0}A_{1/2} + \frac{\partial C_1^{(1)}(t)}{\partial B_0}B_{1/2}$$

$$y_{3/2}^{(r)}(t) = \frac{\partial C_1^{(2)}(t)}{\partial A_0}A_{1/2} + \frac{\partial C_1^{(2)}(t)}{\partial B_0}B_{1/2}$$

$$x_2^{(r)}(t) = A_2\cos kt + \frac{B_2}{k}\sin kt + C_2^{(1)}(t) + \frac{\partial C_1^{(1)}(t)}{\partial A_0}A_1 + \frac{\partial C_1^{(1)}(t)}{\partial B_0}B_1$$

$$y_2^{(r)}(t) = C_2^{(2)}(t) + \frac{\partial C_1^{(2)}(t)}{\partial A_0}A_1 + \frac{\partial C_1^{(2)}(t)}{\partial B_0}B_1 \quad \text{etc.}$$

Here and in what follows the superscript $(r)$ on y $A_{n/2}$ and $B_{n/2}$ will be omitted. Let us investigate the stability of these periodic solutions.

**2.** Let us write down the variational equations for system (1.1)

$$u^{(1)\cdot\cdot} + k^2u^{(1)} = \mu\,(F_x^{(1)}u^{(1)} + F_{x^\cdot}^{(1)}u^{(1)\cdot} + F_y^{(1)}u^{(2)} + F_{y^\cdot}^{(1)}u^{(2)\cdot})_r$$

$$u^{(2)\cdot\cdot} + \omega^2u^{(2)} = \mu\,(F_x^{(2)}u^{(1)} + F_{x^\cdot}^{(2)}u^{(1)\cdot} + F_y^{(2)}u^{(2)} + F_{y^\cdot}^{(2)}u^{(2)\cdot})_r \tag{2.1}$$

The subscript $r$ signifies that in the derivatives of the functions $F^{(1)}$ and $F^{(2)}$, in place of the $x$, $x^\cdot$, $y$, $y^\cdot$ we must substitute the solution of (1.1) from Formulas (1.11).

For an approximate computation of the characteristic exponents we shall use the method presented in [2] (pp.203-213).

Let us first note the coefficients $F_x^{(i)}, F_{x^\cdot}^{(i)}, \ldots$ $(i=1,\,2)$ in Equations (2.1) are analytic functions of $\mu$, $x$, $x^\cdot$, $y$, $y^\cdot$, and the latter, in turn, as solutions of system (1.1), are analytic functions of $\mu$ or $\mu^{\frac{1}{2}}$. Consequently, $F_x^{(i)},\ldots$ are analytic functions of $\mu$ or $\mu^{\frac{1}{2}}$ (see [3 and 4]). For example, with due regard to (1.11)

$$F_x^{(i)}(t, x, x^\cdot, y, y^\cdot, \mu) = (F_x^{(i)})_0 + (F_{xx}^{(i)}x_{1/2} + F_{xx^\cdot}^{(i)}x_{1/2}^\cdot + F_{xy}^{(i)}y_{1/2} + F_{xy^\cdot}^{(i)}y_{1/2}^\cdot)_0\,\mu^{1/2} +$$
$$+ (F_{xx}^{(i)}x_1 + F_{xx^\cdot}^{(i)}x_1^\cdot + F_{xy}^{(i)}y_1 + F_{xy^\cdot}^{(i)}y_1^\cdot + \ldots)_0\,\mu + \ldots \tag{2.2}$$

Similar expansions hold also for the remaining coefficients.

Let us find the characteristic exponents of system (2.1) corresponding to the resonant roots $\pm ik$ of the fundamental equation. When $\mu = 0$ the values of the characteristic exponents are $\alpha_0 = \pm ik$. Since in the resonant case the quantities $\pm ik$ may be rejected from the characteristic exponents, we seek the latter in the form

$$\alpha = \sum_{n=2}^{\infty} \alpha_{n/2}\,\mu^{n/2} \tag{2.3}$$

System (2.1) has the solution

$$u^{(i)}(t) = e^{\alpha t}\,v^{(i)}(t) \tag{2.4}$$

where $v^{(1)}$ and $v^{(2)}$ are periodic functions of period $2\pi$. Consequantly, if in Equations (2.1) we were to substitute the variables (2.4), then the transformed system should admit periodic solution. This condition serves for the determination of the quantity $\alpha_{n/2}$.

We seek the $v^{(1)}$ and $v^{(2)}$ in the form

$$v^{(i)}(t) = \sum_{n=0}^{\infty} v_{n/2}^{(i)}(t)\,\mu^{n/2} \tag{2.5}$$

Let us substitute (2.2), (2.3) and (2.5) into Equations (2.1) after making the substitution (2.4) into them. By equating terms of like powers of $\mu^{\frac{n}{2}}$, for the determination of the functions $v_{n/2}^{(i)}$ we obtain the sequentially-solvable inhomogeneous linear systems of equations

$$v_{n/2}^{(1)\cdot\cdot} + k^2 v_{n/2}^{(1)} = \Phi_{n/2}^{(1)}, \qquad v_{n/2}^{(2)\cdot\cdot} + \omega^2 v_{n/2}^{(2)} = \Phi_{n/2}^{(2)} \tag{2.6}$$

where

$$\Phi_0^{(i)} = 0, \quad \Phi_{1/2}^{(i)} = 0, \quad \Phi_1^{(i)} = -2\alpha_1 v_0^{(i)\cdot} + (F_x^{(i)} v_0^{(1)} + F_{x^\cdot}^{(i)} v_0^{(1)\cdot} + F_y^{(i)} v_0^{(2)} + F_{y^\cdot}^{(i)} v_0^{(2)\cdot})_0$$

$$\begin{aligned}\Phi_{3/2}^{(i)} = & -2\alpha_{1/2} v_0^{(i)\cdot} - 2\alpha_1 v_{1/2}^{(i)\cdot} + (F_x^{(i)} v_{1/2}^{(1)} + F_{x^\cdot}^{(i)} v_{1/2}^{(1)\cdot} + F_y^{(i)} v_{1/2}^{(2)} + F_{y^\cdot}^{(i)} v_{1/2}^{(2)\cdot})_0 + (F_{xx}^{(i)} x_{1/2}^{(r)} + \\ & + F_{xx^\cdot}^{(i)} x_{1/2}^{(r)\cdot} + F_{xy}^{(i)} y_{1/2}^{(r)} + F_{xy^\cdot}^{(i)} y_{1/2}^{(r)\cdot})_0 v_0^{(1)} + (F_{x^\cdot x}^{(i)} x_{1/2}^{(r)} + F_{x^\cdot x^\cdot}^{(i)} x_{1/2}^{(r)\cdot} + F_{x^\cdot y}^{(i)} y_{1/2}^{(r)} + \\ & + F_{x^\cdot y^\cdot}^{(i)} y_{1/2}^{(r)\cdot})_0\, v_0^{(1)\cdot} + (F_{yx}^{(i)} x_{1/2}^{(r)} + F_{yx^\cdot}^{(i)} x_{1/2}^{(r)\cdot} + F_{yy}^{(i)} y_{1/2}^{(r)} + F_{yy^\cdot}^{(i)} y_{1/2}^{(r)\cdot})_0 v_0^{(2)} + \\ & + (F_{y^\cdot x}^{(i)} x_{1/2}^{(r)} + F_{y^\cdot x^\cdot}^{(i)} x_{1/2}^{(r)\cdot} + F_{y^\cdot y}^{(i)} y_{1/2}^{(r)} + F_{y^\cdot y^\cdot}^{(i)} y_{1/2}^{(r)\cdot})_0\, v_0^{(2)\cdot} \quad \text{и т. д.}\end{aligned} \tag{2.7}$$

In what follows we shall also need the values of $\Phi_{n/2}^{(i)}$ up to $n = 5$, but because of their awkwardness we shall not compute them here.

The solutions of (2.6) will be periodic if and only if the conditions

$$\int_0^{2\pi} \Phi_{n/2}^{(1)}(t) \sin kt\, dt = 0, \qquad \int_0^{2\pi} \Phi_{n/2}^{(1)}(t) \cos kt\, dt = 0 \tag{2.8}$$

are satisfied.

The periodic solutions of period $2\pi$ of system (2.6) are

$$v_{n/2}^{(1)}(t) = M_{n/2} \cos kt + \frac{1}{k} N_{n/2} \sin kt + w_{n/2}^{(1)}(t)$$

$$v_{n/2}^{(2)}(t) = w_{n/2}^{(2)}(t) \tag{2.9}$$

Here $M_{n/2}$, $N_{n/2}$ are arbitrary constants, $w_{n/2}^{(1)}(t)$, $w_{n/2}^{(2)}(t)$ is a particular periodic solution of (2.6) of period $2\pi$. For $n = 0.1$ we have, for example, $w_0^{(i)} = w_{1/2}^{(i)} = 0$.

Let us write down the conditions (2.8) for $n = 2$, taking (2.7) and (2.9) into consideration when $n = 0$. Here we must make use of the expressions for the derivatives of $C_1^{(1)}(t)$, $C_1^{(1)\cdot}(t)$ with respect to $A_0$ and $B_0$ when $t = 2\pi$, as computed on the basis of (1.6).

Finally, the stated conditions take the form

$$M_0\left(\frac{\partial C_1^{(1)}}{\partial A_0} - 2\pi\alpha_1\right) + N_0 \frac{\partial C_1^{(1)}}{\partial B_0} = 0, \qquad M_0 \frac{\partial C_1^{(1)\cdot}}{\partial A_0} + N_0\left(\frac{\partial C_1^{(1)\cdot}}{\partial B_0} - 2\pi\alpha_1\right) = 0 \tag{2.10}$$

Here and subsequently when the argument $t$ is omitted from the functions $C_1^{(1)}(t)$, $C_1^{(1)\cdot}(t)$ and their derivatives with respect to $A_0$ and $B_0$, it is to be understood that $t = 2\pi$.

The system of equations (2.10) define the unknown constants $M_0$ and $N_0$. In order that this system have a nontrivial solution it is necessary and sufficient that the determinant in the unkown vanish. By expanding this determinant we obtain a quadratic equation with respect to $\alpha_1$

$$4\alpha_1^2\pi^2 - 2\pi\left(\frac{\partial C_1^{(1)}}{\partial A_0} + \frac{\partial C_1^{(1)\cdot}}{\partial B_0}\right)\alpha_1 + \Delta^\circ = 0, \qquad \Delta^\circ = \frac{\partial (C_1^{(1)}, C_1^{(1)\cdot})}{\partial (A_0, B_0)} \tag{2.11}$$

which is a special case of Equation (13.6) in [2] (p.210).

For sufficiently small $\mu$ the sign of the real part of $\alpha$ is determined by the sign of the real part of the coefficient of the highest term in expansion (2.3). Therefore, in order that the solution of (1.1) be asymptotically stable it is sufficient that the real parts of $\alpha_1$ be less than zero. From (2.11) we obtain the conditions for the real parts of $\alpha_1$ to be negative in the form

$$\text{a)}\quad \frac{\partial C_1^{(1)}}{\partial A_0} + \frac{\partial C_1^{(1)\cdot}}{\partial B_0} < 0, \qquad \text{b)}\quad \Delta^\circ > 0 \tag{2.12}$$

Conditions (2.12) have already been obtained in [2]. They are similar, respectively, to conditions (14) and (9) of the paper [5] for a system with one degree of freedom.

Let the equations for the fundamental amplitudes have a double root. The second condition in (2.12) is no longer satisfied since $\Delta^\circ = 0$. From (2.11) it follows that one of the values of $\alpha_1 = 0$. Let us find the next highest coefficient $\alpha_{n/2}$ of the characteristic exponent, whose real part is different from zero. We consider the most interesting cases similar to that considered in [5].

Here we shall make use of certain quantities obtained from the quantities introduced in [3] by replacing in the latter the $C_n(t)$ of [3] by the $C_n^{(1)}(t)$ of [1]. We shall use the same notations for them except that we shall use a superscript $^\circ$

For example,

$$\Lambda_1^\circ = \frac{\partial C_1^{(1)}}{\partial B_0} C_2^{(1)\cdot} - \frac{\partial C_1^{(1)\cdot}}{\partial B_0} C_2^{(1)}, \qquad \Delta_2^\circ = \frac{\partial C_1^{(1)}}{\partial A_0} C_2^{(1)\cdot} - \frac{\partial C_1^{(1)\cdot}}{\partial A_0} C_2^{(1)} \tag{2.13}$$

1. Let $\Delta_1^\circ \neq 0$. There exist two periodic solutions of (1.1), which can be expanded into series in powere of $\mu^{\frac{1}{2}}$. When $\alpha_1 = 0$, from the Equations (2.10) we have, for example,

$$M_0 = \frac{\partial C_1^{(1)\cdot}}{\partial B_0}, \qquad N_0 = -\frac{\partial C_1^{(1)\cdot}}{\partial A_0} \tag{2.14}$$

For $n = 3$, after cumbersome calculations the conditions (2.8) become

$$M_{1/2}\frac{\partial C_1^{(1)}}{\partial A_0} + N_{1/2}\frac{\partial C_1^{(1)}}{\partial B_0} + W_{1/2}^{(1)} = 0, \qquad M_{1/2}\frac{\partial C_1^{(1)\cdot}}{\partial A_0} + N_{1/2}\frac{\partial C_1^{(1)\cdot}}{\partial B_0} + W_{1/2}^{(2)} = 0 \tag{2.15}$$

where

$$W_{1/2}^{(1)} = A_{1/2}\left(\frac{\partial C_1^{(1)\cdot}}{\partial B_0}\frac{\partial^2 C_1^{(1)}}{\partial A_0^2} - \frac{\partial C_1^{(1)\cdot}}{\partial A_0}\frac{\partial^2 C_1^{(1)}}{\partial A_0 \partial B_0}\right) + B_{1/2}\left(\frac{\partial C_1^{(1)\cdot}}{\partial B_0}\frac{\partial^2 C_1^{(1)}}{\partial A_0 \partial B_0} - \frac{\partial C_1^{(1)\cdot}}{\partial A_0}\frac{\partial^2 C_1^{(1)}}{\partial B_0^2}\right) - 2\alpha_{3/2}\pi\frac{\partial C_1^{(1)\cdot}}{\partial B_0} \tag{2.16}$$

$$W_{1/2}^{(2)} = A_{1/2}\left(\frac{\partial C_1^{(1)\cdot}}{\partial B_0}\frac{\partial^2 C_1^{(1)\cdot}}{\partial A_0^2} - \frac{\partial C_1^{(1)\cdot}}{\partial A_0}\frac{\partial^2 C_1^{(1)\cdot}}{\partial A_0 \partial B_0}\right) + B_{1/2}\left(\frac{\partial C_1^{(1)\cdot}}{\partial B_0}\frac{\partial^2 C_1^{(1)\cdot}}{\partial A_0 \partial B_0} - \frac{\partial C_1^{(1)\cdot}}{\partial A_0}\frac{\partial^2 C_1^{(1)\cdot}}{\partial B_0^2}\right) + 2\pi\alpha_{3/2}\frac{\partial C_1^{(1)\cdot}}{\partial A_0}$$

Let us recall that $A_{n/2}$, $B_{n/2}$ (the coefficients of $\mu^{n/2}$ in expansions (1.9)) have two values [1 and 3] differing in sign and corresponding to two periodic solutions ($r = 1, 2$).

Since the detrminant in the unknowns $M_{1/2}$, $N_{1/2}$ in system (2.15) equals zero, then for this system to be compatible it is necessary and sufficient that one of the Equations (2.15) be a consequence of the other. After some manipulations, from this condition we get the equation for determining $\alpha_{3/2}$

$$L_{5/2}^{\circ} \equiv A_{1/2}\frac{\partial\Delta^{\circ}}{\partial A_0} + B_{1/2}\frac{\partial\Delta^{\circ}}{\partial B_0} = 2\pi\left(\frac{\partial C_1^{(1)}}{\partial A_0} + \frac{\partial C_1^{(1)\cdot}}{\partial B_0}\right)\alpha_{3/2}$$

Since $\alpha_{3/2}$ should be less than zero, and the first condition in (2.12) is assumed as satisfied, then for asymptotic stability it is sufficient to satisfy the condition $L_{5/2}^{\circ} > 0$, similarly, to the condition (10) of [5]. In the same way as there, it can be proved that $L_{5/2}^{\circ} \neq 0$ in the case being considered and that the inequality $L_{5/2}^{\circ} > 0$ is satisfied for one of the two periodic solutions.

2. Let $\Delta_1^{\circ} = 0$, and let the roots of Equation $N_{02}^{\circ}a^2 + N_{11}^{\circ}a + N_{20}^{\circ} = 0$, which is similar to Equation (2.14) of [3], be simple, i.e. $a_1 \neq a_2$. Recall that

$$N_{02}^{\circ} = \frac{1}{2}\left(\frac{\partial C_1^{(1)}}{\partial B_0}\frac{\partial C_1^{(1)\cdot}}{\partial B_0}\right)^{-1}\frac{\partial(C_1'^{(1)\cdot}, \Delta^{\circ})}{\partial(A_0, B_0)}$$

$$\begin{aligned}N_{11}^{\circ} = \left(\frac{\partial C_1^{(1)}}{\partial B_0}\frac{\partial C_1^{(1)\cdot}}{\partial B_0}\right)^{-1}\Bigg[C_2^{\cdot(1)}\Bigg(&\frac{\partial^2 C_1^{(1)\cdot}}{\partial B_0^2}\frac{\partial C_1^{(1)}}{\partial A_0} + \\ &+ \frac{\partial^2 C_1^{(1)}}{\partial A_0\partial B_0}\frac{\partial C_1^{(1)\cdot}}{\partial B_0} - \frac{\partial^2 C_1^{(1)}}{\partial B_0^2}\frac{\partial C_1^{(1)\cdot}}{\partial A_0} - \frac{\partial^2 C_1^{(1)\cdot}}{\partial A_0\partial B_0}\frac{\partial C_1^{(1)}}{\partial B_0}\Bigg) + \\ &+ \frac{\partial C_1^{(1)\cdot}}{\partial B_0}\left(\frac{\partial C_2^{(1)\cdot}}{\partial A_0}\frac{\partial C_1^{(1)}}{\partial B_0} + \frac{\partial C_2^{(1)}}{\partial B_0}\frac{\partial C_1^{(1)\cdot}}{\partial A_0} - \frac{\partial C_2^{(1)\cdot}}{\partial B_0}\frac{\partial C_1^{(1)}}{\partial A_0} - \frac{\partial C_2^{(1)}}{\partial_0}\frac{\partial C_1^{(1)\cdot}}{\partial B_0}\right)\Bigg]\end{aligned}$$

$$\begin{aligned}N_{20}^{\circ} = \left(\frac{\partial C_1^{(1)}}{\partial B_0}\right)^{-2}\left(\frac{\partial C_1^{(1)\cdot}}{\partial B_0}\right)^{-1}\Bigg\{&\frac{\partial C_1^{(1)\cdot}}{\partial B_0}\left[\frac{1}{2}\frac{\partial^2 C_1^{(1)\cdot}}{\partial B_0^2}C_2^{(1)^2} - \right. \\ &\left. - \frac{\partial C_2^{(1)\cdot}}{\partial B_0}\frac{\partial C_1^{(1)}}{\partial B_0}C_2^{(1)} + \left(\frac{\partial C_1^{(1)}}{\partial B_0}\right)^2 C_3^{(1)\cdot}\right] - \frac{\partial C_1^{(1)}}{\partial B_0}\left[\frac{1}{2}\frac{\partial^2 C_1^{(1)}}{\partial B_0^2}C_2^{(1)\cdot 2} - \right. \\ &\left. - \frac{\partial C_2^{(1)}}{\partial B_0}\frac{\partial C_1^{(1)\cdot}}{\partial B_0}C_2^{(1)\cdot} + \left(\frac{\partial C_1^{(1)\cdot}}{\partial B_0}\right)^2 C_3^{(1)}\right]\Bigg\}\end{aligned}$$

In this case there exist two periodic solutions of (1.1), which can be expanded into series in powers of $\mu$. The coefficient $\alpha_{3/2} = 0$, since $L_{5/2}^{\circ} = 0$.

When $\alpha_{3/2} = 0$, $A_{1/2} = B_{1/2} = 0$ for $M_{1/2}$ and $N_{1/2}$ from (2.15) we get the same values as for $M_0$ and $N_0$, respectively, in (2.14). The quantity $\alpha_2$ is found from the existence conditions for periodic solutions of system (2.6) for $n = 4$, which after cumbersome computations are written as

$$\begin{aligned}M_1\frac{\partial C_1^{(1)}}{\partial A_0} + N_1\frac{\partial C_1^{(1)}}{\partial B_0} + W_1^{(1)} + \frac{\partial(C_2^{(1)}, C_1^{(1)\cdot})}{\partial(A_0, B_0)} = 0 \\ M_1\frac{\partial C_1^{(1)\cdot}}{\partial A_0} + N_1\frac{\partial C_1^{(1)\cdot}}{\partial B_0} + W_1^{(2)} + \frac{\partial(C_2^{(1)\cdot}, C_1^{(1)\cdot})}{\partial(A_0, B_0)} = 0\end{aligned} \qquad (2.17)$$

where $W_1^{(1)}$ and $W_1^{(2)}$ are obtained from $W_{1/2}^{(1)}$, and $W_{1/2}^{(2)}$ in (2.16) by replacing in the latter the quantities $A_{1/2}, B_{1/2}, \alpha_{3/2}$ by $A_1, B_1, \alpha_2$, respectively.

Here we have made use of the expression for the periodic solution of (2.6) when $n = 2$, which has the form

$$v_1^{(1)}(t) = M_1\cos kt + \frac{N_1}{k}\sin kt + M_0\frac{\partial C_1^{(1)}(t)}{\partial A_0} + N_0\frac{\partial C_1^{(1)}(t)}{\partial B_0}, \qquad v_1^{(2)}(t) = \ldots \qquad (2.18)$$

and $M_0$ and $N_0$ take the values (2.14).

As in the previous case the compatibility conditions for system (2.17) give the equation for the determination of $\alpha_2$

$$L_3^{\circ} \equiv A_1\frac{\partial\Delta^{\circ}}{\partial A_0} + B_1\frac{\partial\Delta^{\circ}}{\partial B_0} - \left(\frac{\partial\Delta_1^{\circ}}{\partial A_0} - \frac{\partial\Delta_2^{\circ}}{\partial B_0}\right) = 2\pi\left(\frac{\partial C_1^{(1)}}{\partial A_0} + \frac{\partial C_1^{(1)\cdot}}{\partial B_0}\right)\alpha_2$$

By virtue of the same reasons as in [5], $L_3^\circ \neq 0$ since Expression for $L_3^\circ$ is similar to $L_3$ in [5].

On the basis of (2.12) (a), the condition for $\alpha_2$ to be negative, has a form $L_3^\circ > 0$, which is similar to condition (13) in [5].

3. Finally, let $a_1 = a_2$, while the quantity $K^\circ \neq 0$ (similar to $K$ in [5]). We can construct two periodic solutions of (1.1), which can be expanded into series in powers of $\mu^{1/2}$. In this case the quantity $L_3^\circ = 0$ and, therefore, $\alpha_2 = 0$.

The coefficient $\alpha_{5/2}$ is found from the periodicity conditions (2.8) when $n = 5$ in the manner similar to the finding of $\alpha_2$ from (2.8) when $n = 4$. Here we use the value of $v_{3/2}^{(1)}(t)$, as found from (2.6) when $n = 3$. It turns out that

$$v_{3/2}^{(1)}(t) = M_{3/2}\cos kt + \frac{N_{3/2}}{k}\sin kt + M_{1/2}\frac{\partial C_1^{(1)}(t)}{\partial A_0} + N_{1/2}\frac{\partial C_1^{(1)}(t)}{\partial B_0}$$

where $M_{3/2}, N_{3/2}$ are arbitrary constants; $M_{1/2}, N_{1/2}$ have the same values as $M_0$, $N_0$, respectively, in (2.14).

The stated periodicity conditions yield two equations for the determination of $M_{3/2}, N_{3/2}$. Taking into account that $L_3^\circ = 0$, from the compatibility conditions for these equations we obtain the equation for determining $\alpha_{5/2}$

$$L_{7/2}^\circ \equiv A_{3/2}\frac{\partial \Delta^\circ}{\partial A_0} + B_{3/2}\frac{\partial \Delta^\circ}{\partial B_0} = 2\pi\left(\frac{\partial C_1^{(1)}}{\partial A_0} + \frac{\partial C_1^{(1)\cdot}}{\partial B_0}\right)\alpha_{5/2}$$

In a manner similar to that in [5] we can show that $L_{7/2}^\circ \neq 0$, and, consequently, $\alpha_{5/2} \neq 0$. The condition for $\alpha_{5/2}$ to be negative yields the condition for asymptotic stability, $L_{7/2}^\circ > 0$, which is always satisfied for one of the two periodic solutions.

Thus, for the resonance root we obtain results similar to the results for systems with one degree of freedom. The conditions for asymptotic stability, corresponding to the resonance root of the fundamental equation, coincide with the conditions for asymptotic stability of the periodic solutions of a non-self-contained system with one degree of freedom [5]. All that need to be done is to substitute the $C_n(t)$ in the latter with the $C_n^{(1)}(t)$ from [1].

Finally, let us write the stability conditions corresponding to the non-resonance root of the fundamental equation

$$\int_0^{2\pi} (F_{y^\cdot}^{(2)})_0\, dt < 0 \qquad (2.19)$$

This condition is obtained by the same method, however, the characteristic exponent is determined by Formula [2]

$$\alpha = \pm i\omega + \sum_{n=2}^{\infty} \alpha_{n/2}\mu^{n/2}$$

**3.** Let us consider some examples. (1). We take the system of equations from [1]

$$x^{\cdot\cdot} + x = -4\cos 2t + \mu\,[{}^4/_3\, y^\cdot - (1 - x^2)\, x^\cdot]$$

$$y^{\cdot\cdot} + {}^1/_4\, y = 5\cos 2t + \mu\,[{}^4/_3\,(1 - x^2)\, x^\cdot - {}^4/_3\, y^\cdot] \qquad (3.1)$$

The generating solution of (3.1) has the form

$$x_0(t) = A_0\cos t + B_0\sin t - {}^4/_3\cos 2t, \quad y_0(t) = -{}^4/_3\cos 2t$$

The equations for the fundamental amplitudes

$$A_0\,[3 + {}^1/_4\,(A_0^2 + B_0^2)] = 0, \qquad B_0\,[3 + {}^1/_4\,(A_0^2 + B_0^2)] = 0$$

have the solution $A_0 = B_0 = 0$. The periodic solution of (3.1) constructed in [1] is stable since the first stability condition (2.12) is not satisfied.

2). We consider a system of equations of the form

$$x^{\cdot\cdot} + x = \mu\,[\lambda_0\sin t + \alpha\,(1 - x^2)\, x^\cdot + \beta y^\cdot]$$

$$y^{\cdot\cdot} + {}^1/_4\, y = \mu\,[-{}^1/_4\,\lambda_0\sin t + \gamma\,(1 - x^2)\, x^\cdot + \delta y^\cdot] \qquad (3.2)$$

The generating solution depends on two arbitrary constants,

$$x_0(t) = A_0 \cos t + B_0 \sin t, \qquad y_0(t) = 0$$

The equations for the fundamental amplitudes

$$\tfrac{3}{4}\lambda_0 + (6\alpha + 9\gamma)A_0\,[1 - \tfrac{1}{4}(A_0^2 + B_0^2)] + \tfrac{1}{7}A_0\,(9\beta + 12\delta) = 0$$

$$(6\alpha + 9\gamma)B_0\,[1 - \tfrac{1}{4}(A_0^2 + B_0^2)] + \tfrac{1}{7}B_0\,(9\beta + 12\delta) = 0$$

have the solutions $B_0 = 0$ , $A_0 = A$ , where $A$ is the root of the cubic equation

$$(6\alpha + 9\gamma)A^3 + 4\,(6\alpha + 9\gamma + \tfrac{9}{7}\beta + \tfrac{12}{7}\delta)A + 3\lambda_0 = 0$$

Here

$$\Delta^\circ = -\tfrac{3}{16}\,(6\alpha + 9\gamma)A^2 + (6\alpha + 9\gamma)A - (6\alpha + 9\gamma + \tfrac{9}{7}\beta + \tfrac{12}{7}\delta)^2$$

Let us assume that the parameters $\lambda_0, \alpha, \beta, \gamma, \delta$ are such that $\Delta^\circ \neq 0$ . Then we can construct a periodic solution of (3.2) in the form of series in integer powers of $\mu$ . The stability conditions (2.12) for this solution give that $\lambda_0, \alpha, \beta, \gamma, \delta$ should be such that $6\alpha + 9\gamma > 0$ and $\Delta^\circ > 0$ . Condition (2.19) gives $\delta < 0$ .

3). We consider a more interesting example

$$x^{\cdot\cdot} + x = \mu\,(\nu_1 \cos t + \lambda_1 \sin t + c_1 x + \gamma_1 x^3 + d_1 y + g_1 y^3)$$

$$y^{\cdot\cdot} + \tfrac{1}{4}y = \mu\,(\nu_2 \cos t + \lambda_2 \sin t + c_2 x + \gamma_2 x^3 + d_2 y + g_2 y^3 + \delta y^{\cdot}) \tag{3.3}$$

This equation reminds us of the Duffing equation [2] in a quasi-linear formulation. The generating solution has the form

$$x_0(t) = A_0 \cos t + B_0 \sin t, \qquad y_0(t) = 0$$

The equations for the fundamental amplitudes are

$$C_1^{(1)}(2\pi) \equiv \pi\,[\lambda_1 + c_1 B_0 + \tfrac{3}{4}\gamma_1 B_0\,(A_0^2 + B_0^2)] = 0$$

$$C_1^{(1)\cdot}(2\pi) \equiv \pi\,[\nu_1 + c_1 A_0 + \tfrac{3}{4}\gamma_1 A_0\,(A_0^2 + B_0^2)] = 0 \tag{3.4}$$

The condition $\Delta^\circ = 0$ for the multiplicity of the root leads to the following relation between the coefficients of the equations:

$$81\gamma_1(\nu_1^2 + \lambda_1^2) + 16\,c_1^3 = 0 \tag{3.5}$$

The coefficients $c_1$ and $\gamma_1$ should be different signs. Here the roots of Equations (3.4) are

$$A_0 = -\frac{3}{2}\frac{\nu_1}{c_1}, \qquad B_0 = -\frac{3}{2}\frac{\lambda_1}{c_1}$$

The condition for these roots to be double [3] yields

$$\Delta^* = \frac{81}{8}\frac{\gamma_1^2\nu_1^2\lambda_1}{c_1} \neq 0$$

After appropriate computations we obtain

$$C_1^{(1)}(t) = \frac{\nu_1}{16}\left(\frac{1}{3} + \frac{27}{4}\frac{\gamma_1\lambda_1^2}{c_1^3}\right)(\cos t - \cos 3t) - \frac{\lambda_1}{16}\left(\frac{1}{3} + \frac{27}{4}\frac{\gamma_1\nu_1^2}{c_1^3}\right)(3\sin t - \sin 3t)$$

Further, in accordance with (1.10), we compute $\Psi_{11}$ and $\Psi_{21}$ , and then from (1.6) we find $C_1^{(2)}(t)$; substituting $C_1^{(1)}(t)$ and $C_1^{(2)}(t)$ into (1.7) we find $F_2^{(1)}(t)$. In accordance with (1.6), this allows us to find $C_2^{(1)}(t)$ and its derivative. Taking their values for $t = 2\pi$ , we compute $\Delta_1^\circ$ from (2.13)

$$\Delta_1^\circ = -\frac{27\gamma_1\nu_1\pi^2}{8c_1^2}\left\{\frac{2}{3}d_1\left[\nu_2\nu_1 + \lambda_2\lambda_1 + \frac{8c_1^2}{81\gamma_1^2}(3c_2\gamma_1 - c_1\gamma_2)\right] + \right.$$

$$\left. + \frac{45}{32}\frac{\gamma_1\nu_1^2\lambda_1^2}{c_1^2} + \frac{c_1}{144}(7\lambda_1^2 + \nu_1^2) + \frac{9}{128}\frac{\lambda_1^3\gamma_1}{c_1^4}(4c_1^3 + 81\gamma_1\nu_1^2)\right\}$$

It is obvious that by a proper choice of the parameters we can make $\Delta_1^\circ \neq 0$. Then we can construct two periodic solutions of period $2\pi$ of (3.3) which transform into the generating solutions when $\mu = 0$. These solutions are represented as the series in powers of $\mu^{1/2}$ (1.11).

Further, we have [3]

$$A_{1/2} = \pm \left( \frac{2\Delta_1^\circ}{\Delta^*} \frac{\partial C_1^{(1)}}{\partial B_0} \frac{\partial C_1^{(1)}}{\partial B_0} \right)^{1/2} = \pm \left( \frac{243}{32} \frac{\nu_1^2 \gamma_1}{c_1^5} \{\cdots\} \right)^{1/2} \quad B_{1/2} = -\frac{\partial C_1^{(1)} / \partial A_0}{\partial C_1^{(1)} / \partial B_0} A_{1/2} = \frac{\lambda_1}{\nu_1} A_{1/2}$$

The expression within the braces under the radical is the same as that within the braces for $\Delta_1^\circ$ above.

Since $\gamma_1$ and $c_1$ should be different signs, then for $A_{1/2}, B_{1/2}$ to be real it is necessary to satisfy the condition $\{\ldots\} > 0$.

Obviously, when $\gamma_1 \nu_1 < 0$ we have the expression $\Delta_1^\circ > 0$, while when $\gamma_1 \nu_1 < 0$, the expression $\Delta_1^\circ < 0$.

Then, according to (1) and [5], we get the following results: when $\gamma_1 \nu_1 < 0$ the periodic solution corresponding to the plus sign before the radical in $A_{1/2}$ is stable; when $\gamma_1 \nu_1 > 0$ the periodic solution corresponding to the minus sign is stable. In accordance with (2.19) the condition $\delta < 0$ is supplemented to these conditions.

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